

Worcester County Mathematics League

Varsity Meet 1 - October 8, 2025

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 1 - October 8, 2025
Answer Key



Round 1 - Arithmetic

1. $\frac{5}{7}$
2. 2205
3. 39

Round 2 - Algebra I

1. 3
2. 18
3. 9009

Round 3 - Set Theory

1. 17
2. 48
3. 7

Round 4 - Measurement

1. 9
2. 204
3. $(2, 3, 27)$ in exact order

Round 5 - Polynomial Equations

1. $(1, -1)$ in exact order
2. 26
3. 42

Team Round

1. $\frac{9}{200}$
2. 21
3. 15
4. $32\pi - 64$ or $-64 + 32\pi$
5. $1, -\frac{5}{2}$ need both, either order
6. 29929
7. $\frac{3}{4} \pm \sqrt{2}$ or $\frac{3 \pm 4\sqrt{2}}{4}$ or list both values, either order
8. 19
9. $2 + 5i$

Worcester County Mathematics League

Varsity Meet 1 - October 8, 2025

Round 1 - Arithmetic



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Evaluate the expression below, where $2\frac{1}{3}$ and $3\frac{1}{2}$ are mixed numbers. Express your answer as a fraction in the simplified form $\frac{a}{b}$.

$$1 \div \left(2\frac{1}{3}\right) + 1 \div \left(3\frac{1}{2}\right)$$

2. Define $x * y = 2xy + x^y$. Evaluate $(3 * 3) * 2$.

3. In postfix notation, operators and numbers are separated by commas and the operator operates on the two numbers that precede it. For example, $(8, 7, +)$ (postfix notation) is the same as $8 + 7$. Expressions with multiple operations are evaluated by applying operators from left to right. Each operator operates on the two preceding numbers, either or both of which might be the result of previous operations. For example, $(8, 7, +, 9, \cdot)$ (postfix) is the same as $(8 + 7) \cdot 9$, and $(8, 7, 9, +, \cdot)$ is the same as $8 \cdot (7 + 9)$.

If $m = (1, 2, 3, 4, 5, +, +, \cdot, \cdot)$ and $n = (1, 2, 3, 4, 5, \cdot, \cdot, +, +)$, find $|m - n|$.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. $|m - n| =$ _____

St. Peter Marian, Algonquin, AMSA

Worcester County Mathematics League
Varsity Meet 1 - October 8, 2025
Round 2 - Algebra I



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve for x :

$$5 - (3 - 2x) - 5x = 3x - 2(5 + x)$$

2. Several years ago, your math team coach required the team to come to practice on Tuesday and Thursday for two consecutive weeks in October. The sum of the dates (specifically, the day numbers in October; 1 for October 1, 31 for October 31, etc.) of the four practices is divisible by 9 and the sum of the dates of the first and last practices is by 5. What is the date of the first of these four practices?

3. Find the sum of the cubes of the integers on the interval $[-15, 17]$.

ANSWERS

(1 pt) 1. $x =$ _____

(2 pts) 2. _____

(3 pts) 3. _____

Clinton, Quaboag, Clinton

NO CALCULATORS ALLOWED

- $$\begin{aligned} I_{a,b} &= \{x : x \text{ is an integer and } a \leq x \leq b\}. \\ A \uplus B &= \text{the sum of the elements of } A \cup B. \\ A \cap B &= \text{the product of the elements of } A \cap B. \\ n &= I_{1,2} \uplus I_{-1,1} \end{aligned}$$

ANSWERS

(3 pts) 3. _____

Assabet Valley, Shrewsbury, Quaboag

Worcester County Mathematics League

Varsity Meet 1 - October 8, 2025

Round 4 - Measurement



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. A rectangular classroom is designed to have 150ft^3 per occupant to provide enough air for each person in the room. If the dimensions of the classroom are 20 ft X 30 ft X h , what is the minimum height h , in feet, to provide sufficient volume for 36 people?
2. Find the area, in cm^2 , of a trapezoid whose parallel sides measure 10cm and 24cm, and whose lateral sides measure 15cm and 13cm.
3. A sphere fits snugly inside a hollow right triangular prism whose bases are equilateral triangles. Thus, the sphere is tangent to both bases and the three lateral sides of the prism. The ratio of the volume of the sphere to the volume of the prism can be expressed in simplest form as $\frac{(a\sqrt{b})\pi}{c}$ where a , b , and c are integers. Find the ordered triple (a, b, c) .

ANSWERS

(1 pt) 1. _____ ft

(2 pts) 2. _____ cm^2

(3 pts) 3. $(a, b, c) = (\text{_____})$

Shrewsbury, Quaboag, QSC

Worcester County Mathematics League

Varsity Meet 1 - October 8, 2025

Round 5 - Polynomial Equations



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. The graph of $y = x^2 - 2x$ is a parabola. Find the coordinates (x, y) of the vertex of this parabola.

2. If r and s are real numbers and the equation $2x^2 + rx + s = 0$ has one root equal to $3 + 2i$, find s .

3. Find the sum of the squares of the roots of:

$$x^4 + 2x^3 - 19x^2 + 6x + 72 = 0$$

ANSWERS

(1 pt) 1. $(x, y) = (\rule{1.5cm}{0.4pt})$

(2 pts) 2. $s = \rule{2cm}{0.4pt}$

(3 pts) 3. $\rule{2cm}{0.4pt}$

Bromfield, Bromfield, Tantasqua

Worcester County Mathematics League
Varsity Meet 1 - October 8, 2025
Team Round

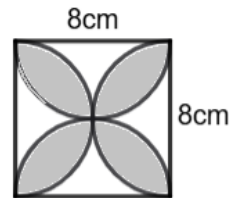


All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Let $a * b = \frac{a}{b^2}$. Find the product of $3 * 4$ and $6 * 5$. Express your answer as a fraction in simplest terms.
2. In a group of cows and chickens, the number of legs is 42 more than twice the number of heads. How many cows are in the group?
3. Records at Lamocow College show that, out of a group of 100 freshmen, 42 are enrolled in Calculus 1, 46 are enrolled in Spanish 1, and 22 are enrolled in Economics 1. Also, 4 are enrolled in both Economics and Spanish, 15 in Calculus and Economics, 9 in Calculus and Spanish, and 12 are enrolled in Calculus and Economics but *not* enrolled in Spanish. How many of the group of 100 students are not enrolled in any of these three courses?

4. Four semicircles lie within a square with side length 8cm, as shown in the figure on the right. Each side of the square is a diameter of one of the four semicircles. Find the sum of the overlapping areas between the four pairs of adjacent semicircles, in cm^2 . That is, find the shaded area shown in the figure. Express your answer in terms of π ; do not approximate.



5. Given polynomial $q(x) = (x - 2k)(x - k)$, where the remainder equals 6 when $q(x)$ is divided by $x + 1$, find all possible values of k .
6. Find the value of:

$$\left[(10^2 - 4) + 16 \div 4 + 24 + (-7)^2 \right]^2$$

7. Solve for x :

$$x^2 - \frac{3}{2}x - \frac{23}{16} = 0$$

8. Given sets A and B where the universal set $U = \{x : x \text{ is an integer and } 0 < x \leq 40\}$ and:

$$A = \{\text{even integers}\},$$

$$B = \{\text{integers that are multiples of 3}\},$$

A' is the complement of A , and likewise for B' : find the number of elements in $(A' \cap B') \cup (A \cap B)$.

9. If $x^2 + 21 - 20i = 0$, where $i = \sqrt{-1}$ and the real part of x is positive, find x and write your answer in the form $a + bi$.

Worcester County Mathematics League
Varsity Meet 1 - October 8, 2025
Team Round Answer Sheet



ANSWERS

1. _____

2. _____

3. _____

4. _____ cm^2

5. $k \in \{ \text{_____} \}$

6. _____

7. $x \in \{ \text{_____} \}$

8. _____

9. $x = \text{_____}$

Worcester County Mathematics League
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7. $\frac{3}{4} \pm \sqrt{2}$ or $\frac{3 \pm 4\sqrt{2}}{4}$ or list both values, either order
8. 19
9. $2 + 5i$

Round 1 - Arithmetic

1. Evaluate the expression below, where $2\frac{1}{3}$ and $3\frac{1}{2}$ are mixed numbers. Express your answer as a fraction in the simplified form $\frac{a}{b}$.

$$1 \div \left(2\frac{1}{3}\right) + 1 \div \left(3\frac{1}{2}\right)$$

Solution: Recall that $1 \div x$ is the reciprocal of a number x , that is, $\frac{1}{x}$. Convert the mixed numbers to fractions, apply the reciprocal rule for fractions $\left(\frac{1}{\frac{a}{b}} = \frac{b}{a}\right)$ and combine the fractions:

$$\begin{aligned} 1 \div \left(2\frac{1}{3}\right) + 1 \div \left(3\frac{1}{2}\right) &= \frac{1}{2 \cdot \frac{3}{3} + \frac{1}{3}} + \frac{1}{3 \cdot \frac{2}{2} + \frac{1}{2}} \\ &= \frac{1}{\frac{6+1}{3}} + \frac{1}{\frac{6+1}{2}} \\ &= \frac{1}{\frac{7}{3}} + \frac{1}{\frac{7}{2}} = \frac{3}{7} + \frac{2}{7} \\ &= \frac{3+2}{7} = \boxed{\frac{5}{7}} \end{aligned}$$

2. Define $x * y = 2xy + x^y$. Evaluate $(3 * 3) * 2$.

Solution: First evaluate $3 * 3 = 2(3)(3) + 3^3 = 6 \cdot 3 + 3 \cdot 3 \cdot 3 = 18 + 27 = 45$. Then evaluate $(3 * 3) * 2 = 45 * 2 = 2(45)(2) + (45)^2 = 180 + 45^2$.

The square of 45 can be calculated by direct multiplication. A slightly faster solution is to apply an identity for finding the square of any integer ending in 5: $(k5)^2 = k(k+1) \cdot 100 + 25$. In this case, $k = 4$, $k(k+1) = 4 \cdot 5 = 20$, and $(45)^2 = 20 \cdot 100 + 25 = 2025$.

Finally, the answer is $(3 * 3) * 2 = 45 * 2 = 180 + 2025 = \boxed{2205}$.

3. In postfix notation, operators and numbers are separated by commas and the operator operates on the two numbers that precede it. For example, $(8, 7, +)$ (postfix notation) is the same as $8 + 7$. Expressions with multiple operations are evaluated by applying operators from left to right. Each operator operates on the two preceding numbers, either or both of which might be the result of previous operations. For example, $(8, 7, +, 9, \cdot)$ (postfix) is the same as $(8 + 7) \cdot 9$, and $(8, 7, 9, +, \cdot)$ is the same as $8 \cdot (7 + 9)$.

If $m = (1, 2, 3, 4, 5, +, +, \cdot, \cdot)$ and $n = (1, 2, 3, 4, 5, \cdot, \cdot, +, +)$, find $|m - n|$.

Solution: Apply parentheses to clarify the order of operations. Then apply the postfix operations, inside to outside, to find m :

$$\begin{aligned} m &= 1, 2, 3, 4, 5, +, +, \cdot, \cdot = 1, (2, (3, (4, 5, +), +), \cdot), \cdot \\ &= 1, (2, (3, (4 + 5), +), \cdot), \cdot = 1, (2, (3, 9, +), \cdot), \cdot \\ &= 1, (2, (3 + 9), \cdot), \cdot = 1, (2, 12, \cdot), \cdot \\ &= 1, (2 \cdot 12), \cdot = 1, 24, \cdot = 1 \cdot 24 = 24 \end{aligned}$$

Proceed likewise to find n :

$$\begin{aligned} n &= 1, 2, 3, 4, 5, \cdot, \cdot, +, + = 1, (2, (3, (4, 5, \cdot), \cdot), +), + \\ &= 1, (2, (3, (4 \cdot 5), \cdot), +), + = 1, (2, (3, 20, \cdot), +), + \\ &= 1, (2, (3 \cdot 20), +), + = 1, (2, 60, +), + \\ &= 1, (2 + 60), + = 1, 62, + = 1 + 62 = 63 \end{aligned}$$

Then $|m - n| = |24 - 63| = |-39| = \boxed{39}$.

Round 2 - Algebra I

1. Solve for
- x
- :

$$5 - (3 - 2x) - 5x = 3x - 2(5 + x)$$

Solution: First distribute the negative sign and the multiplication over the expressions in parentheses, then simplify by collecting terms:

$$\begin{aligned} 5 - (3 - 2x) - 5x &= 3x - 2(5 + x) \\ 5 - 3 - (-2x) - 5x &= 3x - 2(5) - 2x \\ 2 + 2x - 5x &= 3x - 2x - 10 \\ 2 - 3x &= x - 10 \end{aligned}$$

Next, add $3x + 10$ to both sides to isolate x on the right hand side.

$$\begin{aligned} 2 + 10 &= x + 3x \\ 12 &= 4x \end{aligned}$$

and $x = \frac{12}{4} = \boxed{3}$.

2. Several years ago, your math team coach required the team to come to practice on Tuesday and Thursday for two consecutive weeks in October. The sum of the dates (specifically, the day numbers in October; 1 for October 1, 31 for October 31, etc.) of the four practices is divisible by 9 and the sum of the dates of the first and last practices is by 5. What is the date of the first of these four practices?

Solution: Let the date of the first Tuesday equal x . Then the dates of the first Thursday, the second Tuesday, and the second Thursday are $x + 2$, $x + 7$, and $x + 9$, respectively. The sum of these four numbers, $x + x + 2 + x + 7 + x + 9 = x + x + x + x + 2 + 7 + 9 = 4x + 18$ must be divisible by 9.

Let $m|n$ denote “ m divides n ”, or equivalently “ n is divisible by m ”, for the remainder of this solution. Since $9|18$ and $9|(4x + 18)$, it must be true that $9|(4x)$ and also $9|x$. In other words, x is a multiple of 9. All four dates must be less than 31 (the number of days in October), so $x + 9 < 31$ and $x < 31 - 9 = 22$. Thus x must equal 9 or 18, the only two multiples of 9 less than 22.

Next, the sum of the dates of the first Tuesday and last Thursday is $x + (x + 9) = 2x + 9$, and it is given that $5|(2x + 9)$. If $x = 9$, then $2x + 9 = 2(9) + 9 = 27$. Since 27 is not divisible by 5, $x \neq 9$. On the other hand, $2(18) + 9 = 36 + 9 = 45$ and $5|45$, so $x = \boxed{18}$.

3. Find the sum of the cubes of the integers on the interval $[-15, 17]$.

Solution: The notation $[m, n]$ (the closed interval from m to n) represents the set that includes m and n and all numbers (integers, in this case) between m and n . Here $[-15, 17] = \{-15, -14, -13, \dots, 13, 14, 15, 16, 17\}$. Recall that $(-x)^3 = -x^3$. Therefore $(-15)^3 + (15)^3 = (-14)^3 + (14)^3 = 0$; all the negative cubes are cancelled by positive cubes. To find the sum of the cubes on the given interval, we only need to sum the cubes of the last two terms: $16^3 + 17^3$.

The answer can be found by direct calculation: $16 \cdot 16 \cdot 16 + 17 \cdot 17 \cdot 17$. Alternatively, some knowledge of powers of 2 together with some algebra identities can be used to simplify this calculation. Recall that $16 = 2^4$. Then $16^3 = (2^4)^3 = 2^{4 \cdot 3} = 2^{12}$, which can be found by doubling eleven times starting with 2. Alternatively, note that $2^{12} = 2^{10} \cdot 2^2 = 4 \cdot 2^{10}$ and recall that $2^{10} = 1024$. Then $16^3 = 4 \cdot 1024 = 4096$.

Next, use the cube identity $(m + 1)^3 = m^3 + 3m^2 + 3m + 1$ to calculate $(17)^3 = (16 + 1)^3 = (16)^3 + 3 \cdot (16)^2 + 3 \cdot 16 + 1$, recalling that $(16)^2 = 256$:

$$\begin{aligned} (16)^3 + (17)^3 &= (16)^3 + (16)^3 + 3 \cdot (16)^2 + 3 \cdot (16) \\ &= 2 \cdot (16)^3 + 3 \cdot 256 + 3 \cdot 16 + 1 \\ &= 2 \cdot 4096 + 768 + 48 + 1 \\ &= 8192 + 768 + 49 = 8960 + 49 = \boxed{9009} \end{aligned}$$

Round 3 - Set Theory

1. In a group of 100 students: 42 like cake, 71 like ice cream, and 4 like neither cake nor ice cream. How many of those students like both cake and ice cream?

Solution: Let A be the set of students who like cake and B be the set of students who like ice cream. Then the set of students who like both cake and ice cream is $A \cap B$, the intersection of A and B . Note that the number of the students who like either cake, ice cream, or both ($A \cup B$) is equal to $100 - 4 = 96$ students (the total number of students minus the number of students who like neither).

Recall the principle of inclusion-exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$, where $|A|$ indicates the number of elements of set A . Then:

$$\begin{aligned} |A \cup B| = 96 &= |A| + |B| - |A \cap B| \\ &= 42 + 71 - |A \cap B| = 113 - |A \cap B|. \end{aligned}$$

Solve the equation: $96 = 113 - |A \cap B|$ and $|A \cap B| = 113 - 96 = \boxed{17}$.

2. How many subsets of the set $\{a, b, c, d, e, f\}$ contain either element a , element b , or both?

Solution: Call the given set S , that is, $S = \{a, b, c, d, e, f\}$. Recall that a set with n elements will have 2^n subsets. Therefore S has $2^6 = 64$ subsets because it has 6 elements. Let x equal the number of subsets of S that contain either element a , element b , or both. Let y equal the number of subsets of S that contain neither a nor b . Observe that the number of subsets of S is equal to $x + y$ and $x = 64 - y$. Also, observe that y is equal to the number of subsets of $\{c, d, e, f\}$ (S with a and b removed). Therefore $y = 2^4 = 16$ because $\{c, d, e, f\}$ has 4 elements. To conclude, $x = 64 - y = 64 - 16 = \boxed{48}$.

3. Define:

$$I_{a,b} = \{x : x \text{ is an integer and } a \leq x \leq b\}.$$

$$A \uplus B = \text{the sum of the elements of } A \cup B.$$

$$A \cap B = \text{the product of the elements of } A \cap B.$$

$$n = I_{1,2} \uplus I_{-1,1}$$

Find the value of $(I_{1,4} \cap I_{3,5}) \cap I_{1,5} + (I_{n,4} \uplus I_{-5,2})$

Solution: Begin by calculating n , as good a place as any. Expand the interval expressions to find their union, then sum the elements of that union. $I_{1,2} \cup I_{-1,1} = \{1, 2\} \cup \{-1, 0, 1\} = \{-1, 0, 1, 2\}$, and $I_{1,2} \uplus I_{-1,1} = -1 + 0 + 1 + 2 = 2$: $n = 2$.

Next, find the two intervals within the parentheses and find their intersection: $(I_{1,4} \cap I_{3,5}) = \{1, 2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}$. Then proceed with this method, expanding intervals, performing set operations and calculating sums and products:

$$\begin{aligned} \{3, 4\} \cap I_{1,5} + I_{2,4} \uplus I_{-5,2} &= \{3, 4\} \cap \{1, 2, 3, 4, 5\} + \{2, 3, 4\} \uplus \{-5, -4, -3, -2, -1, 0, 1, 2\} \\ &= 3 \cdot 4 + (-5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 + 3 + 4) \\ &= 12 + (-5) = \boxed{7}. \end{aligned}$$

Round 4 - Measurement

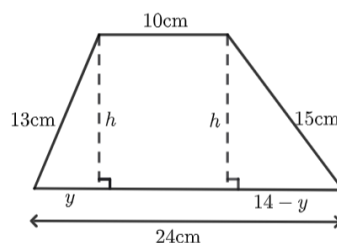
1. A rectangular classroom is designed to have 150ft^3 (cubic feet) per occupant to provide enough air for each person in the room. If the dimensions of the classroom are 20 ft X 30 ft X h , what is the minimum height h to have sufficient volume for 36 people?

Solution: Let h be the height of the room. Then the volume of the room is $V = 20 \cdot 30 \cdot h = 600h$ ft^3 . This volume must be at least $36 \cdot 150\text{ft}^3$ to satisfy the requirement for air per person. Set these two expressions equal: $600h = 36 \cdot 150$. Solve for h :

$$h = \frac{36 \cdot 150}{600} = \frac{36 \cdot 150}{6 \cdot 100} = \frac{36}{6} \frac{150}{100} = 6 \cdot \frac{3}{2} = \boxed{9}\text{ft}$$

2. Find the area, in cm^2 , of a trapezoid whose parallel sides measure 10cm and 24cm, and whose lateral sides measure 15cm and 13cm.

Solution: Recall that the area of a trapezoid is given by: $\frac{h(b_1+b_2)}{2}$, where h is the height of the trapezoid, (the length of its altitude) and b_1 and b_2 are the lengths of the bases, given as the lateral side lengths. It remains to find h . Draw the trapezoid, label dimensions and construct two altitudes drawn from the endpoints of the shorter base, shown at right.



Solution: (continued) Note that the altitudes divide the triangle into a rectangle and two right triangles, each of which has a side (two sides for the rectangle) whose length equals h . The sum of the lengths of the two bottom sides (legs) of the triangles is $24 - 10 = 14\text{cm}$ because, together with the bottom side of the rectangle, they form the 24cm base of the trapezoid. Let the length of the bottom leg of the left triangle equal y . Then the length of the bottom leg of the right triangle equals $14 - y$. Now apply the Pythagorean Theorem to the two right triangles:

$$h^2 + (14 - y)^2 = 15^2$$

$$h^2 + y^2 = 13^2$$

Subtract the second equation from the first to cancel h^2 , and solve for y :

$$(14 - y)^2 - y^2 = 15^2 - 13^2$$

$$14^2 - 28y + y^2 - y^2 = 225 - 169$$

$$196 - 28y = 56$$

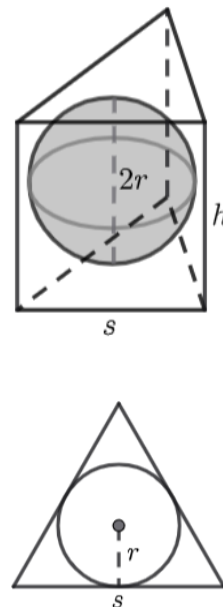
$$196 - 56 = 140 = 28y$$

and $28y = 140$, or $y = \frac{140}{28} = 5$. Then $h^2 = 13^2 - 5^2 = 169 - 25 = 144$, and $h = 12$. Finally, apply the area formula for a trapezoid to find the area: $\frac{12(10+24)}{2} = \frac{12 \cdot 34}{2} = 12 \cdot 17 = \boxed{204}\text{cm}^2$.

3. A sphere fits snugly inside a hollow right triangular prism whose bases are equilateral triangles: the sphere is tangent to both bases and the three lateral sides of the prism. The ratio of the volume of the sphere to the volume of the prism can be expressed in simplest form as $\frac{(a\sqrt{b})\pi}{c}$ where a , b , and c are integers. Find the ordered triple (a, b, c) .

Solution: Sketch the triangular prism with the inscribed sphere, as shown at right. Let r be the radius of the sphere, h be the height of the prism, and s be the length of one side of a base. Recall that the bases of a prism lie in parallel planes. Recall also that, in a right prism, a line segment that joins corresponding vertices of the bases is perpendicular to the two planes. Construct a line segment whose endpoints are the points of tangency of the sphere with the two bases. This line segment is perpendicular to the two bases and its length is therefore h . It is also a diameter of the sphere of length $2r$, so $h = 2r$.

Next, find a relation between s and r . Picture a plane parallel and halfway between the prism bases. It divides the sphere into two congruent hemispheres and the prism into two congruent prisms. The intersection of the plane with the sphere is a circle with radius r , and its intersection with the prism is an equilateral triangle with side length s . Note that the points of tangency of the sphere with the three sides of the prism will lie in this plane. Therefore the circle is inscribed in the triangle, as shown in the figure at the right.



Solution: (continued) Recall that the area of any triangle is equal to its semiperimeter times the radius of a circle inscribed in the triangle. In this case, the semiperimeter is $(s + s + s)/2 = \frac{3}{2}s$ and the radius is r , so another expression for the triangle's area is $\frac{3}{2}rs$. Set these two expressions equal and multiply both sides by $\frac{4}{s\sqrt{3}}$:

$$\frac{\sqrt{3}}{4}s^2 = \frac{3}{2}rs; \quad s = \frac{4}{s\sqrt{3}} \frac{3}{2}rs = 2r\sqrt{3}$$

The volume of a prism is Bh , where B is the area of the base and h is the height of the prism. In this case, the base is an equilateral triangle. Recall that the area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ and substitute the above expression for B :

$$B = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(2r\sqrt{3})^2 = \frac{\sqrt{3}}{4}4r^2 \cdot 3 = 3\sqrt{3}r^2$$

Substitute the expressions for B and h and $V_{\text{prism}} = Bh = 3\sqrt{3}r^2 \cdot 2r = 6\sqrt{3}r^3$. Recall that the volume of a sphere of radius r is $V_{\text{sphere}} = \frac{4\pi}{3}r^3$ and calculate the ratio of volumes:

$$\frac{V_{\text{sphere}}}{V_{\text{prism}}} = \frac{\frac{4\pi}{3}r^3}{6\sqrt{3}r^3} = \frac{4\pi}{18\sqrt{3}} = \frac{2\pi\sqrt{3}}{9(\sqrt{3})^2} = \frac{2\pi\sqrt{3}}{27}$$

Therefore the ordered triple $(a, b, c) = \boxed{(2, 3, 27)}$.

Round 5 - Polynomial Equations

1. The graph of $y = x^2 - 2x$ is a parabola. Find the coordinates (x, y) of the vertex of this parabola.

Solution: Note that the parabola will open “up” because the sign of x^2 is positive. Therefore the vertex will be below the x -axis and $y < 0$. Recall that the vertex of a parabola lies on its axis of symmetry. Set $y = 0$ and factor the quadratic: $y = 0 = x(x - 2)$. Note that the graph intersects the y -axis at $x = 0$ and $x = 2$. Recall that when a quadratic function is graphed, the axis of symmetry of the parabola is vertical (perpendicular to the x -axis) and has the equation $x = k$. By symmetry, it will bisect the line segment between the two x intercepts; in this case, $k = \frac{0+2}{2} = 1$. Recall that the vertex lies on the axis of symmetry. In this case, its x -coordinate equals $x = k = 1$. Plug $x = 1$ into the quadratic to find the y -coordinate of the vertex: $y = (1)^2 - 2(1) = 1 - 2 = -1$. Therefore $(x, y) = \boxed{(1, -1)}$.

2. If r and s are real numbers, and the equation $2x^2 + rx + s = 0$ has one root equal to $3 + 2i$, find s .

Solution: Recall that the roots of polynomials with real coefficients come in complex conjugate pairs. Thus, the two roots are $3 \pm 2i$. Recall Vieta's theorem, which states in part: given quadratic equation $ax^2 + bx + c = 0$, the product of its roots equal $-\frac{c}{a}$. The product of the roots of the polynomial in question is $(3 + 2i)(3 - 2i) = 3^2 - 2^2i^2 = 9 - (-4) = 9 + 4 = 13 = -\frac{s}{2}$. Therefore $s = 2 \cdot 13 = \boxed{26}$.

3. Find the sum of the squares of the roots of:

$$x^4 + 2x^3 - 19x^2 + 6x + 72 = 0$$

Solution: This fourth degree polynomial has four roots by the fundamental theorem of algebra. Let r_1, r_2, r_3, r_4 represent its (not necessarily distinct) four roots. Then $x^4 + 2x^3 - 19x^2 + 6x + 72 = (x - r_1)(x - r_2)(x - r_3)(x - r_4) = 0$. Expand the factored polynomial and group terms by powers of x :

$$\begin{aligned} (x - r_1)(x - r_2)(x - r_3)(x - r_4) &= x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 \\ &\quad - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4 \\ &= x^4 + 2x^3 - 19x^2 + 6x + 72 \end{aligned}$$

Note that this equation is an instance of Vieta's theorem. Next, equate the coefficients of x^3 and x^2 to the corresponding expressions in the expanded polynomial:

$$\begin{aligned} -r_1 - r_2 - r_3 - r_4 &= 2 \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= -19 \end{aligned}$$

Square both sides of the first equation, expand the left hand side, replace the sum of root pair products with -19 , and solve for the requested sum of squares:

$$\begin{aligned} (-r_1 - r_2 - r_3 - r_4)^2 &= (r_1 + r_2 + r_3 + r_4)^2 = 2^2 \\ r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) &= 4 \\ r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2(-19) &= 4 \\ r_1^2 + r_2^2 + r_3^2 + r_4^2 &= 4 + 38 = \boxed{42} \end{aligned}$$

Alternate Solution: Recall Newton's Sums. If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ with roots $r_1, r_2, \dots, r_n$, and S_k is the sum of the k^{th} powers of the n roots, then the following system of equations can be used to solve for S_k :

$$\begin{aligned} a_nS_1 + a_{n-1} &= 0 \\ a_nS_2 + a_{n-1}S_1 + 2a_{n-2} &= 0 \\ &\vdots \\ a_nS_k + a_{n-1}S_{k-1} + \dots + k \cdot a_{n-k} &= 0 \end{aligned}$$

For this problem, only two equations are needed to solve for S_2 :

$$\begin{aligned} S_1 + 2 &= 0 \\ S_2 + 2S_1 + 2 \cdot (-19) &= 0 \end{aligned}$$

Then $S_1 = -2$ and $S_2 + 2(-2) - 38 = 0$, and $S_2 = 4 + 38 = 42$, as above.

Team Round

1. Let $a * b = \frac{a}{b^2}$. Find the product of $3 * 4$ and $6 * 5$. Express your answer as a fraction in simplest terms.

Solution: Evaluate each operation, then multiply the two results:

$$\begin{aligned} 3 * 4 &= \frac{3}{4^2} = \frac{3}{16} \\ 6 * 5 &= \frac{6}{5} = \frac{6}{5^2} = \frac{6}{25} \\ 3 * 4 \cdot 6 * 5 &= \frac{3 \cdot 3 \cdot 2}{2 \cdot 8 \cdot 25} = \boxed{\frac{9}{200}} \end{aligned}$$

2. In a group of cows and chickens, the number of legs is 42 more than twice the number of heads. How many cows are in the group?

Solution: Let h equal the number of heads and let l equal the number of legs in the group. It is given that

$$l = 2h + 42.$$

Let w equal the number of cows in the group and n equal the number of chickens. Each cow has one head and four legs, each chicken has one head and two legs, so $h = w + n$ and $l = 4w + 2n$. Substitute these expressions for h and l in the first equation:

$$\begin{aligned} 4w + 2n &= 2(w + n) + 42 \\ &= 2w + 2n + 42 \\ 2w &= 42 \end{aligned}$$

where $2w + 2n$ was subtracted from both sides to eliminate n and reduce w . Finally, solve for $w = \frac{42}{2} = \boxed{21}$.

3. Records at Lamocow College show that, out of a group of 100 freshmen, 42 are enrolled in Calculus, 46 are enrolled in Spanish, and 22 are enrolled in Economics. Also, 4 are enrolled in both Economics and Spanish, 15 in Calculus and Economics, 9 in Calculus and Spanish, and 12 are enrolled in Calculus and Economics but *not* enrolled in Spanish. How many of the group of 100 students are not enrolled in any of these three courses?

Solution: Note that if 15 students are enrolled in Calculus and Economics and 12 are enrolled in Calculus and Economics but *not* Spanish, then the other $15 - 12 = 3$ students must be enrolled in Calculus, Economics, *and* Spanish. Let C be the set of students enrolled in Calculus, E be the set of students enrolled in Economics, and S be the set of students enrolled in Spanish. Then $|C \cap E \cap S| = 3$, where $|A|$ denotes the number of elements in set A . Also, it was given that $|C \cap E \cap \bar{S}| = 12$, where $\bar{A}$ denotes the set of elements *not* in set A .

Draw a Venn Diagram, shown at left, with the seven subsets of $C \cup E \cup S$ labeled as shown. This will be used to guide the placement of numbers of elements that are calculated for the seven regions of the diagram. Place 3 in the intersection of the three circles because $|C \cap E \cap S| = 3$. Place 12 in the region above this central region because $|C \cap E \cap \bar{S}| = 12$.

Similarly, note that $|S \cap C \cap \bar{E}| = |S \cap C| - |C \cap E \cap S| = 9 - 3 = 6$ and $|S \cap E \cap \bar{C}| = |S \cap E| - |C \cap E \cap S| = 4 - 3 = 1$. Place 6 and 1 in the appropriate regions of the diagram.

Note that the number of elements in C , 42, equals the number of elements in $C \cap \bar{E} \cap \bar{S}$ plus the number of elements in the three other regions of the “ C ” circle, or $|C| = |C \cap \bar{E} \cap \bar{S}| + |C \cap E \cap \bar{S}| + |C \cap \bar{E} \cap S| + |C \cap E \cap S|$. Then $|C \cap \bar{E} \cap \bar{S}| = 42 - (12 + 3 + 6) = 42 - 21 = 21$. Place 21 in the appropriate region.

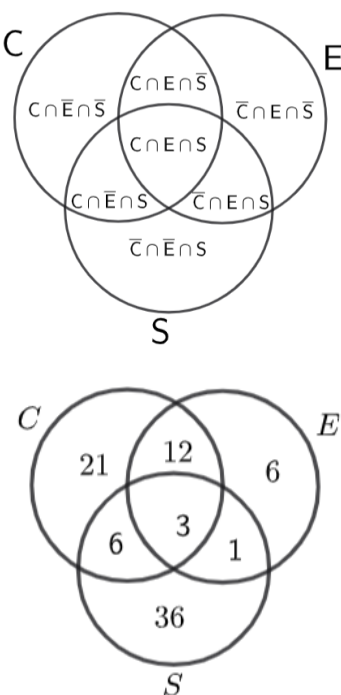
Likewise, $|C \cap \bar{E} \cap \bar{S}| = |E| - 12 - 3 - 1 = 22 - 16 = 6$ and $|S \cap \bar{C} \cap \bar{E}| = |S| - 6 - 3 - 1 = 46 - 10 = 36$. Place 6 and 36 in the appropriate regions.

Finally, total all the numbers in the Venn Diagram and subtract this sum from 100 to find the number of students not enrolled in any of the three courses: $100 - 21 - 12 - 6 - 6 - 3 - 1 - 36 = 100 - 33 - 15 - 37 = 100 - 70 - 15 = 30 - 15 = 15$.

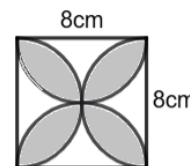
Alternative Solution: Apply the principle of inclusion-exclusion to calculate $|C \cup E \cup S|$, the number of students taking Calculus, Economics, or Spanish. Use the formula for three sets, substituting the given values and the calculated value of $|C \cap E \cap S|$:

$$\begin{aligned} |C \cup E \cup S| &= |C| + |E| + |S| - |C \cap E| - |C \cap S| - |E \cap S| + |C \cap E \cap S| \\ &= 42 + 22 + 46 - 15 - 9 - 4 + 3 = 110 - 28 + 3 \\ &= 85 \end{aligned}$$

Then $|\bar{C} \cap \bar{E} \cap \bar{S}| = 100 - 85 = 15$ as above.

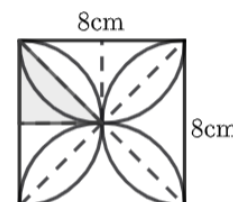


4. Four semicircles lie within a square with side length 8cm, as shown in the figure on the right. Each side of the square is a diameter of one of the four semicircles. Find the sum of the overlapping areas between the four pairs of adjacent semicircles, in cm^2 . That is, find the shaded area shown in the figure. Express your answer in terms of π ; do not approximate.



Solution: Note the symmetry of the shaded region in the original figure of the problem statement. It is composed of four congruent leaf shaped regions. Sketch the diagonals of the square and note that they divide the four leaves exactly in half, where the diagonals are shown as dotted lines in the figure below and to the right. Thus, the original shaded area is equal to eight times the area of one of the half leaves.

Note that each quarter circle (half a semicircle) in the diagram is divided by a diagonal into an isoscles right triangle and a half leaf. One of the triangles is lightly shaded in the solution figure to help visualization. Let A_q be the area of a quarter circle, A_t be the area of a right triangle, and $A_{1/8}$ be the area of a half leaf. Then $A_{1/8} + A_t = A_q$ and $A_{1/8} = A_q - A_t$.



Next, $A_q = \frac{1}{4}\pi r^2$, where $r = 4\text{cm}$ is the radius of the quarter circle; and $A_t = \frac{1}{2}s^2$ where $s = 4\text{cm}$ is the length of the legs of the right triangle. Substitute 4 for s and r :

$$\begin{aligned} A_{1/8} &= A_q - A_t = \frac{1}{4}\pi r^2 - \frac{1}{2}s^2 = \frac{1}{4}\pi 4^2 - \frac{1}{2}4^2 \\ &= 4\pi - 8. \end{aligned}$$

Multiply this area (of a half leaf) by 8 to find the area of the original shaded region: $8A_{1/8} = 8(4\pi - 8) = \boxed{32\pi - 64}$.

5. Given polynomial $q(x) = (x - 2k)(x - k)$, where the remainder equals 6 when $q(x)$ is divided by $x + 1$, find all possible values of k .

Solution: First expand $q(x) = (x - 2k)(x - k) = x^2 - 3kx + 2k^2$. Next, use synthetic division to divide this polynomial by $x + 1$:

$$\begin{array}{r|rrrr} -1 & 1 & -3k & 2k^2 & \\ & & -1 & 3k+1 & \\ \hline & 1 & -3k-1 & 2k^2+3k+1 & \end{array}$$

Set this remainder equal to 6: $2k^2 + 3k + 1 = 6$, or $2k^2 + 3k - 5 = 0$. Factor this quadratic: $(2k + 5)(k - 1) = 0$ and $2k + 5 = 0$ or $k - 1 = 0$; therefore $k \in \left\{ -\frac{5}{2}, 1 \right\}$.

6. Find the value of:

$$\left[(10^2 - 4) + 16 \div 4 + 24 + (-7)^2\right]^2$$

Solution: Evaluate the expression using the PEMDAS rule for order of operations. First, $10^2 - 4 = 100 - 4 = 96$ and $(-7)^2 = (-7)(-7) = 49$. (parentheses first, exponentials before multiplication, division, addition, or subtraction). Next, $16 \div 4 = 4$, so that the expression inside the brackets evaluates to: $(10^2 - 4) + 16 \div 4 + 24 + (-7)^2 = 96 + 4 + 24 + 49 = 100 + 73 = 173$. Then calculate $173^2 = (100 + 70 + 3) \cdot 173 = 100 \cdot 173 + 70 \cdot 173 + 3 \cdot 173 = 17300 + 12110 + 519 = 29410 + 519 = \boxed{29929}$.

7. Solve for x :

$$x^2 - \frac{3}{2}x - \frac{23}{16} = 0$$

Solution: First, multiply the equation by 16 to create an equivalent equation with integer coefficients: $16(x^2 - \frac{3}{2}x - \frac{23}{16}) = 16x^2 - 24x - 23 = 0$. Now that the quadratic is in standard form ($ax^2 + bx + c = 0$), calculate the discriminant $b^2 - 4ac$: $(-24)^2 - (-4)(16)(23) = (3 \cdot 8)^2 + 64 \cdot 23 = 3^2 \cdot 8^2 + 23 \cdot 64 = 9 \cdot 64 + 23 \cdot 64 = (9 + 23)64 = 32 \cdot 64 = 2^5 \cdot 2^6 = 2^{5+6} = 2^{11}$. The discriminant is not a perfect square and so the quadratic is not factorable. Instead, apply the quadratic equation, using the discriminant that you just calculated:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-24) \pm \sqrt{2^{11}}}{2(16)} \\ &= \frac{24 \pm \sqrt{2^{10}}\sqrt{2^1}}{32} = \frac{24 \pm 2^5\sqrt{2}}{32} \\ &= \frac{3 \cdot 8 \pm 32\sqrt{2}}{32} = \frac{3 \pm 4\sqrt{2}}{4} = \frac{3}{4} \pm \sqrt{2} \end{aligned}$$

8. Given sets A and B where the universal set $U = \{x : x \text{ is an integer and } 0 < x \leq 40\}$ and:

$$A = \{\text{even integers}\},$$

$$B = \{\text{integers that are multiples of 3}\},$$

A' is the complement of A , and likewise for B' : find the number of elements in $(A' \cap B') \cup (A \cap B)$.

Solution: Apply the definitions of A , B , and U to determine that $A \cap B$ is the set of even multiples of 3 between 1 and 40. Its elements must be multiples of 6, so $A \cap B = \{6, 12, \dots, 36\}$, with 6 elements. The complement of set A is the set of elements in U not in set A . Therefore A' is the set of odd numbers, B' is the set of numbers not divisible by 3, and $A' \cap B'$ is the set of odd positive numbers less than 41 that are not divisible by 3. Thus, $A' \cap B' = \{1, 5, 7, 11, 13, \dots, 31, 35, 37\}$. The number of elements of $A' \cap B'$ (denoted $|A' \cap B'|$) can be found directly by simply counting them, or it can be found by noting that it is the union of two arithmetic series: $A' \cap B' = \{1, 7, 13, \dots, 31, 37\} \cup \{5, 11, \dots, 29, 35\}$ with 7 and 6 elements, respectively. Therefore $|A' \cap B'| = 7 + 6 = 13$ because the two series share no common numbers (they are disjoint). Finally, note that $A \cap B$ and $A' \cap B'$ are disjoint, so that $|(A' \cap B') \cup (A \cap B)| = |A \cap B| + |A' \cap B'| = 6 + 13 = \boxed{19}$.

9. If $x^2 + 21 - 20i = 0$, where $i = \sqrt{-1}$ and the real part of x is positive, find x and write your answer in the form $a + bi$.

Solution: Let $x = a + bi$, where a and b are Real numbers. Then $x^2 = (a + bi)^2 = a^2 + 2abi + (bi)^2 = a^2 + 2abi + b^2i^2 = a^2 - b^2 + 2abi$. Rewrite the given polynomial to isolate x^2 : $x^2 = -21 + 20i = a^2 - b^2 + 2abi$. Given the two sides of the last equation, set the real and imaginary parts equal to form a system of two equations in a and b :

$$-21 = a^2 - b^2$$

$$20 = 2ab$$

Solve the second equation for b : $b = \frac{10}{a}$. Substitute this expression for b in the first equation:

$$\begin{aligned} -21 &= a^2 - \left(\frac{10}{a}\right)^2 \\ &= a^2 - \frac{100}{a^2}. \end{aligned}$$

Add 21 to both sides of the equation to group polynomial terms together, multiply by a^2 , and factor the resulting polynomial in a^2 :

$$\begin{aligned} 0 &= (a^2)^2 + 21a^2 - 100 \\ &= (a^2 - 4)(a^2 + 25) \end{aligned}$$

Note that $a^2 + 25 = 0$ has no real solution, so $a^2 - 4 = 0$, $a^2 = 4$, and $a = \pm 2$. Then $b = \frac{10}{a} = \frac{10}{\pm 2} = \pm 5$ and $a + bi = 2 + 5i$ or $-2 - 5i$. The question specifies that the real part is positive. Reject the second solution and $a + bi = \boxed{2 + 5i}$.